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# Toward the evaluation of $P(X_{(t)} > Y_{(t)})$ when both $X_{(t)}$ and $Y_{(t)}$ are inactivity times of two systems

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## ABSTRACT

The inactivity time, also known as reversed residual life, has been a topic of increasing interest in the literature. In this investigation, based on the comparison of inactivity times of two devices, we introduce and study a new estimate of the probability of the inactivity time of one device exceeding that of another device. The problem studied in this paper is important for engineers and system designers. It would enable them to compare the inactivity times of the products and, hence to design better products. Several properties of this probability are established. Connections between the target probability and the reversed hazard rates of the two devices are established. In addition, some of the reliability properties of the new concept are investigated extending the well-known probability ordering. Finally, to illustrate the introduced concepts, many examples and applications in the context of reliability theory are included.

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Bias sampling; mean inactivity time; reliability; reliability ordering; reversed hazard rate;  $U$ -statistics.

## 1. Introduction and related work

The inactivity time has recently become a topic of great importance in reliability theory and life testing applications (see, e.g., Ortega, 2009; Khanjari, 2008; Finkelstein, 2002; Kalbfleish and Lawless, 1991; Ahmad, Kayid, and Pellerey, 2005; Ahmad and Kayed, 2005; Eryilmaz, 2010; Goliforsushani et al., 2012; Zardasht and Asadi, 2010; Chandra and Roy, 2001; Block, Savits, and Singh, 1998; Li and Lu, 2003; Nanda et al., 2003, among others). Even if the inactivity time has been mainly used in reliability, it has been useful to describe the behavior of lifetime random variables in survival retrospective studies (Andersen et al., 1993, and some applications have been derived in the risk theory), and econometrics (Eeckhoudt and Gollier, 1995; Kijima and Ohnishi, 1999; and Mi, 1999). An epidemiological research is concerned with both, the instant of infections and the time elapsed since the moment till the time of observation, that is, the mean inactivity time (MIT).

Based on the inactivity time function, various types of stochastic orders and associated properties have been developed rapidly over the years, resulting in a large body of literature (see, e.g., Nanda et al., 2003; Kayid and Ahmad, 2004; Ahmad, Kayid, and Pellerey, 2005; and Li and Xu, 2006).

Let  $X$  be a random variable with probability density function and distribution function given, respectively, denoted by  $f(x)$  and  $F(x)$ . Its reversed hazard rate (RHR) is defined as

$$\tau_X(x) = \frac{f(x)}{F(x)}$$

The MIT is given by

$$m(t) = E[t - X | X \leq t] = \frac{\int_0^t F(u) du}{F(t)}$$

Block, Savits, and Singh (1998) presented some properties of the RHR along with the affinity to study parallel systems and reversed rate ordering in  $k$ -out-of- $n$  systems. Finkelstein (2002) considers the application of RHR and MIT to ordering of random variables with proportional RHR model.

Chandra and Roy (2001) pointed out the growing importance of the RHR and analyzed the relationship with respect to the monotonic behavior between the RHR and MIT, presenting characterization properties.

Di Creszeno (2000) presented some interesting results on the proportional reversed hazard model concerning aging characteristics and stochastic order.

Zardasht and Asadi (2010) studied the problem of estimating the probability that the mean residual life (MRL) of a random variable  $X$  exceeds the corresponding MRL of a random variable  $Y$ .

## 1.1. Preliminaries

Zardasht and Asadi (2010) have considered the problem of estimating the probability that the residual lifetime of a component whose lifetime is  $X$  exceeds that of a component whose lifetime is  $Y$ . The dual concept of inactivity time has not yet been addressed in the literature. Thus, the aim of this paper is to fill this gap in the literature.

To this end, we consider two systems with lifetimes  $X$  and  $Y$  and assume that they have failed at time  $t > 0$ . We raise the following question: what is the probability that, at time  $t$ , the inactivity time  $X_{(t)}$  be greater than the inactivity time  $Y_{(t)}$ ? We denote this probability by  $\eta(t)$ , that is,  $\eta(t) = P(X_{(t)} > Y_{(t)})$ .

This article focuses on the study of  $\eta(t)$ , providing many of its statistical and reliability properties. The study of the properties of  $\eta(t)$  might be important for engineers and system designers to compare the inactivity times of the products and, hence to design better products.

This paper is organized as follows. Section 2 obtains the form of  $\eta(t)$  in terms of distribution functions  $F$  and  $G$ . Survival properties of  $\eta(t)$  are derived in this section. It is shown that when the ratio of the RHRs of  $X$  and  $Y$  is a monotone function of time, then  $\eta(t)$  is also monotone function of time. It is proved that under the condition that the ratio of the RHRs of  $X$  and  $Y$  is known,  $\eta(t)$  uniquely determines the distribution function  $F$  (and hence the distribution functions of  $G$ ). In Section 3, we define the concept of inactivity probability ordering extending the probability ordering introduced by Mi (1999). In the other words, if for all values of  $t$ ,  $\eta(t) \leq 0.5$ , we say that the system with lifetime  $Y$  is better than the system with lifetime  $X$  in inactivity probability. It is proved in this section that, when the failure rate of  $Y$  is less than the reversed failure rate of  $X$ , at time  $t$ , then  $\eta(t) \leq 0.5$ . The closure property of the proposed concept concerning the operation of the mixture of probability distribution and the operation of the random minima is also studied. Finally, in Section 3, we show that under some mild condition  $\eta(t) \leq 0.5$  implies that  $X$  and  $Y$  are stochastically

ordered. Section 4 is devoted to the estimation of  $\eta(t)$ , using the concept of  $U$ -statistics and obtains the asymptotic distribution on the estimator.

### 2. Some properties of $\eta(t)$

Let  $X$  and  $Y$  be two independent non negative continuous random variables with survival functions  $F$  and  $G$ , respectively. Let  $X_{(t)} = t - X|X < t$  and  $Y_{(t)} = t - Y|Y < t$  be their respective inactivity times. Observe that

$$\begin{aligned} \eta(t) &= P(X_{(t)} > Y_{(t)}) = P(t - X > t - Y|X < t, Y < t) \\ &= \frac{\int_0^t F(x) dG(x)}{F(t)G(t)} = 1 - \frac{\int_0^t G(x) dF(x)}{F(t)G(t)} \end{aligned} \tag{1}$$

provided that  $F(t) > 0$  and  $G(t) > 0$ .

Clearly,  $\eta(t)$ , as a function, measures the probability that the inactivity time of the system with lifetime  $X$  is greater than the inactivity time of the system with lifetime  $Y$  at time  $t$ .

Some examples are given below.

**Example 1.** Let  $X$  and  $Y$  follow Weibull distributions with distribution functions

$$F(x) = 1 - e^{-(x/\beta_1)^{\alpha_1}}, \quad x > 0, \alpha_1 > 0, \beta_1 > 0$$

and

$$G(x) = 1 - e^{-(x/\beta_2)^{\alpha_2}}, \quad x > 0, \alpha_2 > 0, \beta_2 > 0$$

respectively. Then

$$\eta(t) = \frac{\alpha_2 \beta_2^{-\alpha_2} \int_0^t x^{\alpha_2-1} \left(1 - e^{-(x/\beta_1)^{\alpha_1}}\right) e^{-(x/\beta_2)^{\alpha_2}} dx}{\left(1 - e^{-(t/\beta_1)^{\alpha_1}}\right) \left(1 - e^{-(t/\beta_2)^{\alpha_2}}\right)}$$

Figure 1 shows the graphs of  $\eta(t)$  for different values of  $(\alpha_1, \beta_1)$  and  $(\alpha_2, \beta_2)$ .

One can see, from Figure 1, that when both distributions are Weibull, with different combinations of the parameters one may have a constant  $\eta(t)$  or it may be increasing (decreasing) for low or moderate values of  $t$  but converges to a constant as  $t$  tends to infinity.

For our next result, let  $\rho(t) = \tau_Y(t)/\tau_X(t)$ , for all  $t$  for which the ratio is well defined.

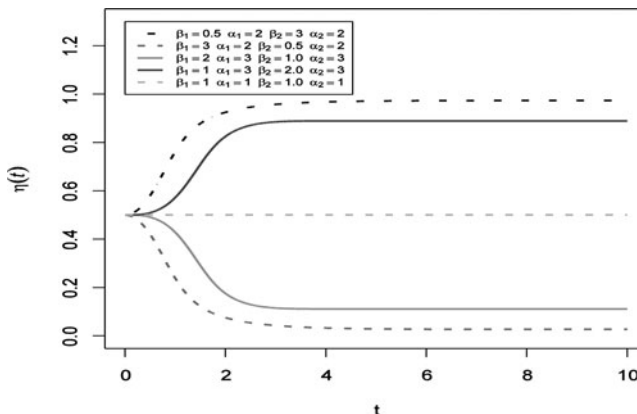


Figure 1. Graphs of  $\eta(t)$  for the Weibull distributions in Example 1.

**Theorem 1.** (1)  $\eta(t)$  is increasing (decreasing) if  $\rho(t)$  is increasing (decreasing),  $\forall t \in \text{supp}_\eta$ .  
 (2)  $\eta(t)$  is constant if  $\rho(t)$  is constant  $\forall t$ . (3)  $\eta(t)$  is a bathtub shaped with at most change points  $\tau^* < \tau_0$ , if  $\rho(t)$  is bathtub shape (upside-down bathtub) with change point  $t_0$ .

**Proof.** First note that  $\eta(t)$  can be written as

$$\eta(t) = \frac{\int_0^t F(x) dG(x)}{F(t)G(t)} = \frac{\int_0^t F(x)G(x)\tau_Y(x)dx}{F(t)G(t)} = \frac{\int_0^t u(x)f_m(x)dx}{F(t)G(t)}$$

where  $u(x) = \frac{\rho(x)}{1+\rho(x)}$  and  $f_m(x)$  is the density corresponding to  $\max(X, Y)$ .

Since  $x/(1+x)$  is strictly increasing in  $x > 0$ , it can be concluded that the shape of  $\rho(x)$  is the same as that of  $u(x)$ . Thus, it is enough to show that the shape of  $u(t)$  determines that of  $\eta(t)$ . On the other hand, it is easy to see that the derivative of  $\eta(t)$  is equal to

$$\eta'(t) = \frac{f_m(t) \int_0^t [u(t) - u(x)] f_m(x) dx}{[F(t)G(t)]^2} \quad (2)$$

The behavior of  $\eta'(t)$  depends on the quantity  $[u(t) - u(x)]$  in the integrand. If  $u(x)$  is increasing (decreasing), then  $\eta'(t) \geq (\leq) 0$ . This proves part 1 of the theorem.

To prove part 2, note that, if for all  $x$ ,  $u(x)$  is constant (and consequently  $\rho(t)$  is constant), then the term  $[u(t) - u(x)]$  in Equation (2) is zero in the entire domain of the support. Thus, it follows that  $\eta'(t) = 0$ , that is,  $\eta(t)$  is a constant.

On the other hand, one can also easily see that

$$\eta'(t) = \tau_Y(t) - \eta(t) [\tau_X(t) + \tau_Y(t)] = [u(t) - \eta(t)] [\tau_X(t) + \tau_Y(t)] \quad (3)$$

This implies that when  $\eta(t)$  is constant, then so is  $\rho(t)$ . This completes the proof of part 2.

To prove part 3, assume  $u(x)$  is minimum (maximum) at  $t_0$  and let

$$K(t) = \int_0^t [u(t) - u(x)] f_m(x) dx$$

From Equation (2), it follows that for all  $t \geq t_0$ ,  $K(t) > 0$ , and hence  $\eta'(t) \geq (\leq) 0$ . This means that if  $\eta(t)$  has a minimum (maximum), it happens at a point before  $t_0$ .

On the other hand,

$$K'(t) = u'(t) F(t) G(t)$$

This implies that for all  $t \geq (\leq) t_0$ ,  $K(t)$  is increasing (decreasing).

Thus,  $\eta(t)$  has a minimum (maximum) at  $t^* \geq t_0$ , if  $K(t)$  has a change of sign before  $t_0$ . This also happens if  $P(X > Y) < (>) \frac{g(0)}{g(0)+f(0)}$  when the ratio is well defined. Hence the proof of part 3 is complete.  $\square$

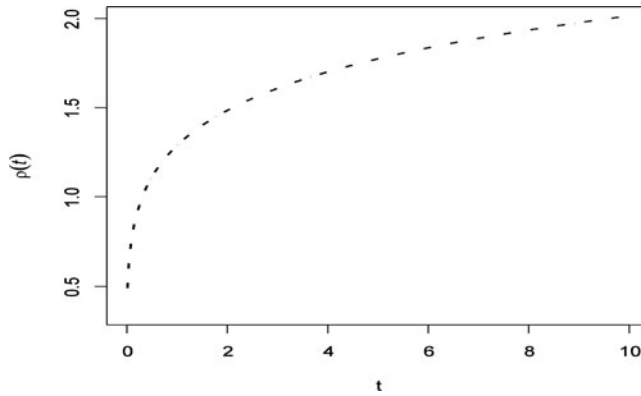
**Remark 1.** From Equation (1) it is easily seen that when  $X$  and  $Y$  are identically distributed, we have  $\eta(t) = 0.5$ . From the proof of Theorem 1, it can be concluded that the converse is also true. In other words, if

$$\eta(t) = \frac{1}{2}, \quad \forall t > 0, \quad \text{then } \eta_X = \eta_Y.$$

The following examples show some applications of Theorem 1.

**Example 2.** Let  $X$  and  $Y$  have, respectively, the distribution functions

$$F(t) = 1 - e^{-t^{0.2}}, \quad t > 0 \quad \text{and} \quad G(t) = 1 - e^{-t^{0.1}}, \quad t > 0$$



**Figure 2.** Plot of  $\rho(t)$  for the distributions in Example 2.

Then, it follows that

$$\rho(t) = \frac{2t^{0.1}(e^{-t^{0.2}} - 1)}{(e^{-4t^{0.1}} - 1)}$$

Figure 2 shows the plots of  $\rho(t)$ . The plot shows that  $\rho(t)$  is increasing in  $t$ .

**Example 3.** Let  $X$  and  $Y$  be exponentially distributed with respective distribution functions

$$F(x) = 1 - e^{-3x}, \quad x > 0 \quad \text{and} \quad G(x) = 1 - e^{-2x}, \quad x > 0$$

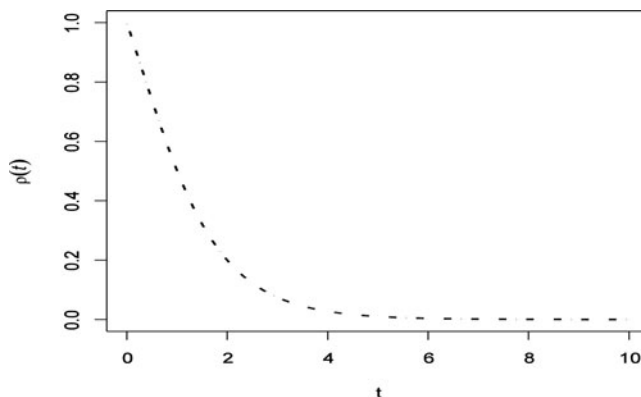
Hence

$$\rho(t) = \frac{3(e^{2t} - 1)}{2(e^{3t} - 1)}$$

Figure 3 shows the plot of  $\rho(t)$  for the distributions in Example 3. The plot shows that  $\rho(t)$  is decreasing in  $t$ .

**Remark 2.** From Equation (3), it can be easily concluded that  $\eta(t)$  is an increasing (decreasing) function of  $t$  if and only if

$$\eta(t) \geq (\leq) \frac{\tau_Y(t)}{\tau_X(t) + \tau_Y(t)}$$



**Figure 3.** Plots of  $\rho(t)$  for the distributions in Example 3.

Now consider a series system with two independent components. If  $X$  and  $Y$  denote the lifetime of the components, then clearly the lifetime of the system is  $T = \min\{X, Y\}$ . Cha and Mi (2007) have considered the probability function  $p(t) = P(Y = T|T = t)$ , that is, the probability that component  $Y$  causes the system failure given that the system fails at time  $t$ .

They showed that  $p(t)$  is given by

$$p(t) = \frac{\tau_Y(t)}{\tau_X(t) + \tau_Y(t)}$$

Thus, one can conclude that  $\eta(t)$  is an increasing (decreasing) function of  $t$  if and only if

$$\eta(t) \geq (\leq) p(t).$$

The following theorem gives some bounds for  $\eta(t)$ , in terms of the distribution functions of  $X$  and  $Y$ , when they are stochastically ordered.

**Theorem 2.** (1) If  $Y \leq_{st} X$ , then  $\eta(t) \leq \frac{G(t)}{2F(t)}$ . (2) If  $X \leq_{st} Y$ , then  $\eta(t) \geq 1 - \frac{F(t)}{2G(t)}$ .

The proof is straightforward and is omitted.

Note that in part (1) the upper bound always lies in the interval  $[0.5, 1]$ , and in part (2) the lower bound lies in the interval  $[0, 0.5]$ .

**Theorem 3.** Let  $X_i$ ,  $i = 1, 2$  and  $Y$  be non negative continuous random variables with distribution function  $F_i$  and  $G$ , respectively. If  $X_1 \leq_{rh} X_2$ , then  $\eta_1(t) \geq \eta_2(t)$ , where  $\eta_i(t) = P(X_i < Y | X_i < t, Y < t)$ ,  $i = 1, 2$ .

**Proof.** The assumption  $X_1 \leq_{rh} X_2$  implies that  $X_{1(t)} \leq_{st} X_{2(t)}$ , where

$$X_{i(t)} = t - X_i | X_i < t, \quad i = 1, 2$$

This implies that

$$\begin{aligned} \eta_1(t) &= \int_0^t F_{1(t)}(x) dG(x) \\ &\geq \int_0^\infty F_{2(t)}(x) dG(x) \\ &= \eta_2(t) \end{aligned}$$

where

$$F_{i(t)}(x) = \frac{F_i(x-t)}{F_i(t)}, \quad i = 1, 2$$

This completes the proof. □

**Corollary 1.** Let  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$  denote the order statistics of a random sample from distribution  $F$ . Also, let

$$\eta_{X_{(i)}}(t) = P(X_{(i)} < Y | X_{(i)} < t, Y < t), \quad i = 1, 2, \dots, n$$

1. One can easily verify that

$$\eta_{X_{(i)}}(t) \leq \eta(t) \leq \eta_{X_{(i+1)}}(t)$$

Hence, based on Theorem 3, we have

$$\eta_{X_{(i)}} \leq \eta(t) \leq \eta_{X_{(i+1)}}$$

2. It is well known that for  $k = 1, 2, \dots, n - 1$ ,  $X_{(k)} \leq_{rh} X_{(k+1)}$  (for details, see, Shaked and Shanthikumar, 2007, p. 31). Hence, based on Theorem 3, we obtain

$$\eta_{X_{(k+1)}} \leq \eta_{X_{(k)}}$$

**Example 4.** Let  $X$  be a continuous non negative random variable with density function  $f$  and a finite mean  $\mu$ . The size-based random variable  $X^*$  corresponding to  $X$  is a random variable with density function

$$f^*(x) = \frac{xf(x)}{\mu}, \quad x > 0$$

(see, Rao, 1997, for applications of size-based distributions). It can be easily seen that the RHR of  $X^*$  is given by

$$\tau^*(x) = \frac{x}{\omega(x)} \tau(x)$$

where  $\omega(x) = E(X|X < x)$ ,  $x > 0$ .

Since  $\omega(x) \leq x$ , we get  $\tau^*(x) \geq \tau(x)$ . Hence, we have  $\eta_{X^*} \leq \eta_X$ .

The following example indicates that the converse of Theorem 3 is not, in general, true.

**Theorem 4.** If  $X$  and  $Y$  are two random variables with finite range, and if  $X$  is IRHR(DRHR) and  $Y$  is DRHR(IRHR), then  $\eta(t)$  is a decreasing (increasing) function of  $t$ .

**Proof.** We assume that  $X$  is IRHR and  $Y$  is DRHR. The same proof can be given for other case. Based on definition,  $X$  is IRHR if and only if, for  $t_1 < t_2$ ,  $X_{(t_2)} \leq_{st} X_{(t_1)}$ , where  $X_{(t)} = t - X|X < t$ . Thus, we have

$$\begin{aligned} \eta(t_1) &= \int_0^{t_1} F_{(t_1)}(x) dG_{(t_1)}(x) \\ &\geq \int_0^{t_2} F_{(t_2)}(x) dG_{(t_1)}(x) \\ &= \int_0^{t_2} G_{(t_1)}(x) dF_{(t_2)}(x) \\ &\geq \int_0^{t_2} G_{(t_2)}(x) dF_{(t_1)}(x) \\ &= \eta(t_2) \end{aligned}$$

□

**Theorem 5.** If  $X$  is NBU and  $Y$  is NWU, then  $\eta(t_1) < \eta(t_2)$ .

This proof is similar to the proof of Theorem 4 and is therefore omitted.

### 3. Probability ordering of residual lifetimes

Mi (1999), for comparing between the lifetime of systems, defined the concept of probability ordering. Let  $X$  and  $Y$  denote the lifetimes of two systems.  $Y$  is said to be greater than  $X$  in probability ordering (denoted by  $X \leq_{pr} Y$ ), if  $P(X > Y) \leq 0.5$ . (For some applications of this



ordering, we refer the reader to Hollander and Samaniego, 2008.) In the following, we extend the concept of probability ordering (pr) to the inactivity probability ordering (ipr) through replacing  $X$  and  $Y$  by the inactivity random variables  $X_{(t)}$  and  $Y_{(t)}$ , respectively.

### 3.1. Definition

Let  $X$  and  $Y$  be non negative continuous random variables denoting the lifetimes of two systems.  $Y$  is said to be greater than  $X$  in inactivity probability (denoted by  $X \leq_{\text{ipr}} Y$ ) if, for all  $t > 0$ ,  $\eta(t) \leq 0.5$ .

The ordering ipr can be applied in some reliability problems. For instance, in burn-in procedures (refer to Block and Savits, 1997) components or systems are subjected to a period of intensive use (or accelerated testing) for a period of time, say  $b > 0$ , before they are released into general usage. Therefore, the lifetime of components that survive the burn-in procedure is actually  $X_{(b)} = b - X | X < b$ . Thus, the ipr ordering can be used for comparison between successfully burned-in products.

**Remark 3.** Note that,  $\eta(t)$  can be written as

$$\eta(t) = \frac{\int_0^t F(x)g(x)dx}{\int_0^t F(x)g(x)dx + \int_0^t G(x)f(x)dx}$$

This implies that  $X \leq_{\text{ipr}} Y$ , if and only if

$$\int_0^t [F(x)g(x) - G(x)f(x)]dx \leq 0$$

The inequality (4) can be written as

$$\int_0^t F(x)G(x)[\tau_Y(x) - \tau_X(x)]dx \leq 0$$

As an immediate conclusion of this inequality is the following.

**Theorem 6.** If  $X \leq_{\text{rhr}} Y$ , then  $X \leq_{\text{ipr}} Y$ .

The following theorem gives a necessary and sufficient condition under which  $X \leq_{\text{ipr}} Y$ .

**Theorem 7.**  $X \leq_{\text{ipr}} Y$ , if and only if  $G(X) \leq_{\text{mit}} G(Y)$

**Proof.** Let  $Z = G(X)$  and denote by  $\alpha_Z(t)$  the MIT function of  $Z$ . Then, we have

$$\eta(t) = \frac{\alpha_Z(G(t))}{G(t)}$$

Now,  $X \leq_{\text{ipr}} Y$  implies  $\eta(t) \leq 0.5$ .

This in fact implies that, for  $t > 0$ ,  $\frac{\alpha(G(t))}{G(t)} \leq 0.5$  or equivalently, for

$$0 < u < 1, \quad \alpha_{Z(u)} = 0.5u.$$

The right-hand side of the last inequality is MIT of a uniform distribution on  $(0, 1)$ . Since,  $G(Y)$  has uniform distribution on  $(0, 1)$ . The proof is complete.  $\square$

Mixture failure populations arise in many fields of applied sciences. For example, in testing the lifetime of systems in the reliability engineering, usually one deals with population that is not homogeneous but rather is mixture of some sub-populations. In a general setting, let  $F_\alpha$  be the distribution function of the sub-populations where  $\alpha$  is assumed to be non negative

random variable with the distribution function  $H(\alpha)$ . Then the survival function of the mixture  $F_\alpha$ , with mixing distribution  $H(\alpha)$ , which we denote by  $F$  is defined as

$$F(x) = \int_0^t F_\alpha(x) dH(\alpha)$$

Now we have the following theorem.

**Theorem 8.** *Let  $X_\alpha$  be distributed as  $F_\alpha$  and  $Y$  be distributed as  $G$ . If  $X_\alpha \leq_{\text{ipr}} Y$ , then  $X \leq_{\text{ipr}} Y$ , where  $X$  is the random variable corresponding to mixture of  $F_\alpha$  with mixing distribution  $H$  on  $\alpha$ , with survival function (5).*

**Proof.** Let  $X_\alpha$  be distributed as  $F_\alpha$  and  $Y$  be distributed as  $G$ . If  $X_\alpha \leq_{\text{ipr}} Y$  implies that

$$\int_0^t [F_\alpha(x)g(x) - G(x)f_\alpha(x)]dx \leq 0,$$

this in turn implies that

$$\begin{aligned} & \int_0^t [F_\alpha(x)g(x) - G(x)f_\alpha(x)] \\ &= \int_0^t \left[ \int_0^\infty (F_\alpha(x)g(x) - G(x)f_\alpha(x))dG(\alpha) \right] dx \\ &= \int_0^\infty \left[ \int_0^t (F_\alpha(x)g(x) - G(x)f_\alpha(x))dx \right] dH(\alpha) \leq 0 \end{aligned}$$

That is,  $X \leq_{\text{ipr}} Y$ . This completes the proof. □

Proportional odds family (also known as tilt parameter family) is a well-known family in the literature (Bennett, 1983; Kirmani and Gupta, 2001; Marshall and Olkin, 2007).

**Definition 1.** Let  $F(x)$  be a distribution function and assume that  $F(x|p)$  is defined in terms of  $F$  as follows

$$\frac{F(x|p)}{\bar{F}(x|p)} = \frac{p\bar{F}(x)}{F(x)}, \quad x > 0, p > 0$$

where  $p$  is called a tilt parameter, and  $F(x|p)$  is called proportional odds family.

It is easily seen that

$$F(x|p) = \frac{F(x)}{p + (1 - p)F(x)}, \quad x > 0, p > 0$$

Now we have the following theorem.

**Theorem 9.** *Let  $X, X^*$ , and  $Y$  be distributed as  $F, F(x|p)$ , and  $G$ , respectively, where  $0 < p < 1$ . If  $X \leq_{\text{ipr}} Y$  then  $X^* \leq_{\text{ipr}} Y$ .*

**Proof.**

$$\begin{aligned} \eta_{X^*Y}(t) &= \frac{\int_0^t F(x|p)/(1 - p)F(x) dG(t)}{F(t)G(t)/p + (1 - p)F(x)} \\ &\leq \frac{\int_0^t F(x)dG(x)}{F(t)G(t)} \\ &= \eta_{XY}(t) \end{aligned}$$

This implies that when  $\eta_{XY}(t) \leq 0.5$ , then  $\eta_{X^*Y}(t) \leq 0.5$ , that is, if  $X \leq_{\text{ipr}} Y$ , then  $X^* \leq_{\text{ipr}} Y$ .  $\square$

**Remark 4.** There are situations where the life length may be considered as the extreme of large number of iid random variables in which the number of variables are random. The distribution function has also another natural derivation as follows: following Marshall and Olkin (2007), let  $X_1, X_2, \dots$  be a sequence of iid random variables with common distribution function  $F$ , and suppose that  $N$  be geometric random variable independent of  $X_i$ 's with probability mass function

$$P(N = n) = (1 - p)^{n-1} p, \quad n = 1, 2, \dots$$

Let  $X^* = \max\{X_1, X_2, \dots, X_N\}$ , then it is easy to see that

$$P(X^* \leq x) = \frac{F(x)}{p + (1 - p)F(x)} = F(x|p)$$

Using this remark and Theorem 9, we get that in the case where  $X_i \leq_{\text{ipr}} Y_i$ , for  $i = 1, 2, \dots$ , then  $X^* \leq_{\text{ipr}} Y$ .

**Theorem 10.** Suppose that  $\eta(t) \geq \eta(0) = P(X > Y)$ ,  $t \geq 0$ . If  $X \leq_{\text{ipr}} Y$ , then  $X \leq_{\text{st}} Y$ .

**Proof.** From the assumption and proof of Theorem 7, one can write

$$\frac{\alpha_Z(G(t))}{G(t)} = \eta(t) \geq \eta(0) = \alpha_Z(0), \quad \text{for all } t \geq 0$$

This is equivalent to

$$\frac{\alpha_Z(u)}{u} \geq \alpha_Z(0), \quad \text{for all } 0 < u < 1$$

Since the MIT of uniform random variable is  $u/2$ , the above inequality is equivalent to

$$\frac{\alpha_Z(t)}{\alpha_u(t)} \geq \frac{\alpha_Z(0)}{\alpha_u(0)}$$

where  $\alpha_Z(t)$  and  $\alpha_U(t)$  are the MIT functions of  $Z = G(X)$  and  $U = G(Y)$ , respectively. On the other hand, from Theorem 7 it follows that  $X \leq_{\text{ipr}} Y$  is equivalent to  $Z \leq_{\text{mit}} U$ . Thus, the result immediately follows from versions of Theorems 1.A.3 and 2.A.3 of Shaked and Shanthikumar (2007).  $\square$

#### 4. Estimation of $\eta(t)$

In this section, we explore a non parametric empirical estimate of  $\eta(t)$ . Note that  $\eta(t)$  can be written as

$$\eta(t) = \frac{\eta_1(t)}{\eta_2(t)}$$

where

$$\eta_1(t) = P(X < t < Y)$$

and

$$\eta_2(t) = P(X < t, Y < t)$$

One can estimate  $\eta(t)$  by replacing  $\eta_1(t)$  and  $\eta_2(t)$  with their corresponding  $U$ -statistics.

$$\hat{\eta}_1(t) = \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n I(X_i < Y_j < t)$$

and

$$\hat{\eta}_2(t) = \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n I(X_i < t, Y_j < t)$$

and manage to show that  $\hat{\eta}(t) \rightarrow \eta(t)$ , in some sense.

## 5. Conclusion

This paper answers the following question: what is the probability that, at time  $t$ , the inactivity time  $X_{(t)}$  be greater than the inactivity time  $Y_{(t)}$ ? We denote this probability by  $\eta(t) = P(X_{(t)} > Y_{(t)})$  And study the properties of  $\eta(t)$ . We obtain the form of  $\eta(t)$  in terms of distribution functions  $F$  and  $G$ . Survival properties of  $\eta(t)$  are derived. It is shown that when the ratio of the RHRs of  $X$  and  $Y$  is monotone function of time, then  $\eta^*(t)$  is also monotone function of time. It is also proved that under the condition that the ratio of the RHRs of  $X$  and  $Y$  is known,  $\eta(t)$  uniquely determines the distribution function  $F$  (and hence the distribution functions of  $G$ ). We define a concept of  $X$  is greater than  $Y$  in inactivity probability extending the concept of probability ordering given by Mi (1999). The closure property of the proposed concept concerning the operation of the mixture of probability distribution and the operation of the random minima are also studied. It is shown that under some mild conditions that  $\eta(t) \leq 0.5$  implies that  $X$  and  $Y$  are stochastically ordered. Finally, we leave an open question about the estimation of  $\eta(t)$  using the concept of  $U$ -statistics. Furthermore, asymptotic properties of the distribution of a derived estimator are another open area that needs to be explored.

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